Rates of convergence of autocorrelation estimates for autoregressive Hilbertian processes

Serge GUILLAS

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Université Paris VI, ISUP-LSTA
Tour 45-55, Boîte 158
4, place Jussieu, 75252 Paris Cedex 05

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Serge Guillas
Université Paris VI (L.S.T.A.) and École des Mines de Douai
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Abstract
We show the consistency in the $L^2$ sense of an estimator of the autocorrelation operator $\rho$ in the autoregressive Hilbertian of order one model $X_n = \rho(X_{n-1}) + \varepsilon_n$. Two main cases are considered, and we obtain upper bounds for the corresponding rates.

1 Introduction

Let $H$ be a real and separable Hilbert space with norm $\| \cdot \|$. Let $\rho$ be a bounded operator on $H$. We suppose that $\sum_{n=0}^{\infty} \| \rho^n \| < \infty$, where $\| \cdot \|_L$ is the linear norm of operators in $H$. Let $(\varepsilon_n)$ be a strong Hilbertian white noise (SWN), that is a sequence of i.i.d. random variables with values in $H$ satisfying

$$\forall n \in \mathbb{Z}, E\varepsilon_n = 0, 0 < E\|\varepsilon_n\|^2 = \sigma^2 < \infty.$$ 

We will consider in this paper the autoregressive Hilbertian of order one model, denoted by $ARH(1)$. It is the unique stationary solution of the equation

$$X_n = \rho(X_{n-1}) + \varepsilon_n. \quad (1)$$

See [1] for an extensive study of the $ARH(1)$ model.

Such Hilbertian processes can theoretically and practically handle situations where continuous-time processes are involved. Precisely, if $(x_t, t \in \mathbb{R})$ is a continuous-time process with continuous paths, then

$$X_k(t) = x_{k\delta+t}, 0 \leq t \leq \delta, k \in \mathbb{Z}$$

is a discrete $L^2([0, \delta])$-valued process. Various applications have already been made. For example, Cavallini et al. (1994) made forecasts of electricity consumption; and, by means of smoothing splines, Besse and Cardot (1996) predicted traffic while Besse et al. (2000) made forecasts of the climatic variation called el niño.

Several extensions of the $ARH(1)$ model have been made. We may mention $ARH(p)$ models - see [8]-, and $ARH(1)$ models with exogenous variables - see [6]. Besides, Cardot et al. (1999) studied a regression model with similar techniques.
Let us denote by $C$ and $D$ respectively the covariance and cross-covariance operator of the stationary process $X$:

$$C(x) = E \langle X_0, x \rangle, D(x) = E \langle X_0, X_1 \rangle.$$ 

It can easily be shown that $C$ is a symmetric positive and compact operator. Defining for all elements $u, v$ in $H$ the operator $u \otimes v$ by

$$u \otimes v(x) = \langle u, x \rangle v, x \in H,$$

we then obtain the decomposition in a complete orthonormal basis $(v_j)$ of $H$:

$$C = \sum_{j=1}^{\infty} \lambda_j v_j \otimes v_j,$$

where $(\lambda_j)$ is a decreasing sequence of positive numbers such that

$$\sum_{j=1}^{\infty} \lambda_j = E \|X_0\|^2 < \infty.$$

The estimation of $\rho$ is a rather intricate problem. Indeed, classical techniques such as maximum likelihood or least squares are not accurate in this Hilbertian context. A technique due to Bosq (2000) works as follows: estimate the eigenvectors $(v_j)$ and the eigenvalues $(\lambda_j)$ of the covariance operator and try to use the relation $D = \rho C$ in order to get $\rho$. $C_n$ and $D_n$ are the following respective unbiased estimators of $D$ and $C$:

$$C_n = \frac{1}{n} \sum_{i=1}^{n} X_i \otimes X_i, D_n = \frac{1}{n-1} \sum_{i=1}^{n-1} X_i \otimes X_{i+1},$$

and we denote by $(v_{jn})$ and $(\lambda_{jn})$ the empirical eigenelements of $C_n$. We would like to define an estimator of $\rho$ as $\rho_n = D_n C_n^{-1}$, but $C_n$ is not invertible in general, so we have to make a projection on the space $H_{k_n}$ spanned by the $k_n$ first eigenvectors of $C_n$, obtaining this way an invertible operator in $H_{k_n}$. Naturally, the choice of $k_n$ may not be easy and is usually done empirically or by a cross validation procedure. In this paper, we will give some ideas about this choice in relatively precise situations.

Bosq (2000) showed almost sure consistency of $\rho_n$. Mas (1999) obtained results about limit in distribution of $\rho_n$. The purpose of this paper is to establish consistency of a slight modification of $\rho_n$ in the $L^2$ mode, that is to say by considering $E \|\hat{\rho}_{n,a} - \rho\|_2^2$, and to obtain rates of convergence when the eigenvectors are known and when they are not.

While in the finite dimensional case this rate of convergence may reach a $\frac{1}{n}$-rate when the eigenvalues of $C$ are bounded by below - see [1, section 8.1] -, we will find in the infinite dimensional case where the eigenvectors are known a $n^{-1/3}$-rate, and in the general case a $n^{-1/4}$-rate.

In both cases, we will assume the existence of a sequence $(a_n)$ satisfying:

- $0 < a_n \leq 1, n \in \mathbb{N}$.
- $\exists 0 < \beta < 1, a_n \leq \beta \lambda_{k_n}, n \in \mathbb{N}$.

We also make use of the following assumptions:

$(H_1)$: $X$ is a $ARH(1)$ such that $E \|X_0\|^4 < \infty$.
$(H_2)$: For all $j$, $\lambda_j > 0$.
$(H_3)$: For all $j$, $\lambda_j > \lambda_{j+1}$. 

2
2 Known eigenvectors

The case considered here is the case where the eigenvectors \( v_j \) of \( C \) are known. Consider the following unbiased estimators of the \( (\lambda_j) \):

\[
\hat{\lambda}_{j,n} = \frac{1}{n} \sum_{i=1}^{n} (X_i, v_j)^2.
\]

For consistency of the \( \hat{\lambda}_{j,n} \), see [1].

Consider now the following estimators of \( C \):

\[
\hat{C}_n = \sum_{j=1}^{k_n} \hat{\lambda}_{jn} v_j \otimes v_j, \quad \hat{C}_{n,a} = \sum_{j=1}^{k_n} \max(\hat{\lambda}_{jn}, a_n) v_j \otimes v_j.
\]

Let us define

\[
\hat{\rho}_{n,a} = \pi^{k_n} D_n \hat{C}^{-1}_{n,a} \pi^{k_n}
\]

where \( \pi^{k_n} \) denotes the orthogonal projector over \( H_{k_n} \), and

\[
\hat{C}^{-1}_{n,a} = \sum_{j=1}^{k_n} \left[ \max(\hat{\lambda}_{jn}, a_n) \right]^{-1} v_j \otimes v_j.
\]

Our goal is to find an upper bound for \( E \| \hat{\rho}_{n,a} - \rho \|^2_F \).

**Lemma 1** Under \((H_1)\) and \((H_2)\),

\[
E \| \hat{\rho}_{n,a} - \rho \|^2_F \leq \frac{c_0}{n a_n^2} + \frac{c_1}{n \lambda_k a_n^2} + \frac{c_2}{n \lambda_k^2 a_n^2} + 2 \lambda_k^2 + 1.
\] (2)

**Proof.** First, write the decomposition

\[
\hat{\rho}_{n,a} - \rho = \left( \pi^{k_n} D_n \hat{C}^{-1}_{n,a} \pi^{k_n} - \pi^{k_n} \rho \pi^{k_n} \right) + \left( \pi^{k_n} \rho \pi^{k_n} - \rho \right).
\]

Observe now that

\[
\pi^{k_n} \rho \pi^{k_n}(x) = \pi^{k_n} D \sum_{j=1}^{k_n} \lambda_j^{-1} (v_j, x) v_j
\]

and let us set

\[
C_{\pi^{k_n}} = \sum_{j=1}^{k_n} \lambda_j v_j \otimes v_j
\]

\[
C^{-1}_{\pi^{k_n}} = \sum_{j=1}^{k_n} \lambda_j^{-1} v_j \otimes v_j.
\]

Accordingly,

\[
\hat{\rho}_{n,a} - \rho = \left( \pi^{k_n} D_n \hat{C}^{-1}_{n,a} \pi^{k_n} - \pi^{k_n} D C^{-1}_{\pi^{k_n}} \pi^{k_n} \right) + \left( \pi^{k_n} \rho \pi^{k_n} - \rho \right).
\] (3)
The first term may be written
\[ \pi^{k_n} D_n \hat{C}_{n,a}^{-1} - \pi^{k_n} D C_{\pi^{k_n}}^{-1} \pi^{k_n} = \pi^{k_n} \left( D_n \hat{C}_{n,a}^{-1} - D C_{\pi^{k_n}}^{-1} \right) \pi^{k_n} \]
\[ = \pi^{k_n} \left[ (D_n - D) \hat{C}_{n,a}^{-1} + D \left( \hat{C}_{n,a}^{-1} - C_{\pi^{k_n}}^{-1} \right) \right] \pi^{k_n} \]
\[ = \pi^{k_n} \left[ (D_n - D) \hat{C}_{n,a}^{-1} - D \hat{C}_{n,a}^{-1} \left( \hat{C}_{n,a} - C_{\pi^{k_n}} \right) C_{\pi^{k_n}}^{-1} \right] \pi^{k_n} \]
hence,
\[ \left\| \pi^{k_n} D_n \hat{C}_{n,a}^{-1} - \pi^{k_n} D C_{\pi^{k_n}}^{-1} \pi^{k_n} \right\|_L^2 \]
\[ \leq 2 \left\| D_n - D \right\|_L^2 \left\| \hat{C}_{n,a}^{-1} \right\|_L^2 + 2 \left\| D \right\|_L^2 \left\| \hat{C}_{n,a}^{-1} \right\|_L \left\| C_{\pi^{k_n}}^{-1} \right\|_L \left\| \hat{C}_{n,a} - C_{\pi^{k_n}} \right\|_L \]
\[ \leq 2a_n^{-2} \left\| D_n - D \right\|_L^2 + 2a_n^{-2} \left\| D \right\|_L^2 \left\| C_{\pi^{k_n}}^{-1} \right\|_L \left\| \hat{C}_{n,a} - C_{\pi^{k_n}} \right\|_L^2 \]
because
\[ \left\| \hat{C}_{n,a}^{-1} \right\|_L \leq a_n^{-1}. \]
Thus, by (3),
\[ E \left\| \hat{\rho}_{n,a} - \rho \right\|_L^2 \leq 2 \left[ 2a_n^{-2} E \left\| D_n - D \right\|_L^2 + 2a_n^{-2} \left\| D \right\|_L^2 \left\| C_{\pi^{k_n}}^{-1} \right\|_L \left\| \hat{C}_{n,a} - C_{\pi^{k_n}} \right\|_L^2 \right] \]
\[ + 2E \left\| \rho \pi^{k_n} - \rho \right\|_L^2. \tag{4} \]
The second term of the right-hand side is easily bounded from above by $2\lambda^2_{k_n+1}$. For the first term \cite[Th. 4.8]{1} gives
\[ E \left\| D_n - D \right\|_L^2 = O \left( \frac{1}{n} \right), \]
and clearly
\[ \left\| C_{\pi^{k_n}}^{-1} \right\|_L^2 = \frac{1}{\lambda^2_{k_n}}. \]
Moreover,
\[ E \left\| \hat{C}_{n,a} - C_{\pi^{k_n}} \right\|_L^2 \leq 2E \left( \left\| \hat{C}_{n,a} - \hat{C}_n \right\|_L 1_{\hat{C}_{n,a} \neq \hat{C}_n} \right) + 2E \left( \left\| \hat{C}_n - C_{\pi^{k_n}} \right\|_L 1_{\hat{C}_{n,a} = \hat{C}_n} \right) \]
\[ + E \left( \left\| \hat{C}_{n,a} - C_{\pi^{k_n}} \right\|_L 1_{\hat{C}_{n,a} = \hat{C}_n} \right). \]
Now, we find an upper bound to $P \left( \hat{C}_{n,a} \neq \hat{C}_n \right)$, knowing that the sequence \( (\hat{\lambda}_{jn}) \) is not decreasing with respect to \( j \). Observe that
\[ P \left( \hat{C}_{n,a} \neq \hat{C}_n \right) = P \left( a_n > \min_{j=1,\ldots,k_n} \hat{\lambda}_{jn} \right). \]
Let us define the discrete random variable \( I_{kn} = \arg \min \{ \hat{\lambda}_{jn}, j = 1, \ldots, k \} \). We then obtain

\[
P \left( \hat{C}_{n,a} \neq \hat{C}_n \right) = P \left( a_n > \hat{\lambda}_{I_{kn}n} \right) \\
= P \left( \hat{\lambda}_{I_{kn}n} - \lambda_{I_{kn}} < a_n - \lambda_{I_{kn}} \right) \\
\leq P \left( \left| \hat{\lambda}_{I_{kn}n} - \lambda_{I_{kn}} \right| \geq (1 - \beta) \lambda_{I_{kn}} \right) \\
\leq P \left( \left| \hat{\lambda}_{I_{kn}n} - \lambda_{I_{kn}} \right| \geq (1 - \beta) \lambda_{kn} \right),
\]

so

\[
P \left( \hat{C}_{n,a} \neq \hat{C}_n \right) \leq P \left( \sup_{j=1, \ldots, k_n} \left| \hat{\lambda}_{jn} - \lambda_j \right| \geq (1 - \beta) \lambda_{kn} \right) \\
\leq \frac{K}{n(1 - \beta)^2 \lambda_{kn}^2}
\]

with a constant \( K > 0 \), applying the Chebychev inequality since

\[
E \left\| \hat{C}_n - C \right\|_L^2 \leq 2E \left\| \hat{C}_n - C_n \right\|_L^2 + 2E \left\| C_n - C \right\|_L^2
\]

and

\[
E \left\| \hat{C}_n - C_n \right\|_L^2 = E \left\| \sum_{j=1}^{k_n} \left[ \hat{\lambda}_{jn} - \lambda_{jn} \right] v_j \otimes v_j \right\|_L^2 \\
\leq E \sup_{j=1, \ldots, k_n} \left| \hat{\lambda}_{jn} - \lambda_{jn} \right|^2 \leq 2E \sup_{j=1, \ldots, k_n} \left| \hat{\lambda}_{jn} - \lambda_j \right|^2 + 2E \sup_{j=1, \ldots, k_n} |\lambda_{jn} - \lambda_j|^2 \\
= O \left( \frac{1}{n} \right)
\]

by \([1, \text{Th. 4.4, Cor 4.5}]\), so by \([1, \text{Th. 4.1}]\)

\[
E \left\| \hat{C}_n - C \right\|_L^2 = O \left( \frac{1}{n} \right).
\]

Note that

\[
\left\| \hat{C}_{n,a} - \hat{C}_n \right\|_L^2 = \left\| \sum_{j=1}^{k_n} \left[ \max \left( \hat{\lambda}_{jn}, a_n \right) - \hat{\lambda}_{jn} \right] v_j \otimes v_j \right\|_L^2 \leq a_n^2
\]

and that

\[
E \left\| \hat{C}_n - C_{\pi, kn} \right\|_L^2 = E \left\| \sum_{j=1}^{k_n} \left[ \hat{\lambda}_{jn} - \lambda_j \right] v_j \otimes v_j \right\|_L^2 \\
\leq E \sup_{j=1, \ldots, k_n} \left| \hat{\lambda}_{jn} - \lambda_j \right|^2 \\
\leq E \left\| C_n - C \right\|_L^2 = O \left( \frac{1}{n} \right),
\]

by [1, Th. 4.4, Cor 4.5], so by [1, Th. 4.1]
Therefore we get, by (4)
\[ E \|\hat{\rho}_{n,a} - \rho\|^2 \leq \frac{c_0}{na_n} + \frac{c_1}{n\lambda_k}\lambda_k^\gamma + \frac{c_2}{n\lambda_k^2 a_n^2} + 2\lambda_k^2. \]

\[ \text{Theorem 1} \] Suppose that \((H_1)\) and \((H_2)\) hold, and that there exist \(\alpha > 0, 0 < \beta < 1, \varepsilon < 1/2\) and \(\gamma \geq 1\) such that
\[ \alpha \frac{\lambda_k^\gamma}{n^{\varepsilon}} \leq a_n \leq \beta \lambda_k, \]
then
\[ E \|\hat{\rho}_{n,a} - \rho\|^2 = O \left( \frac{1}{n(1-2\varepsilon)} \lambda_k^{2(1+\gamma)} \right) + O \left( \lambda_k^2 \right). \]

\[ \text{Proof.} \] It is an easy consequence of (2), using the inequalities \(\alpha \frac{\lambda_k^\gamma}{n^{\varepsilon}} \leq a_n \) and \(\lambda_{k+1} \leq \lambda_k\). ■

\[ \text{Remark 1} \] The optimal choice of \(\lambda_k\) is such that:
\[ \lambda_k^2 = \frac{c}{n(1-2\varepsilon)\lambda_k^{2+2\gamma}}, \text{ i.e. } \lambda_k^{4+2\gamma} = \frac{c}{n(1-2\varepsilon)}, c > 0. \] (5)

The rate of convergence in quadratic mean is then of order
\[ \lambda_k^2 \asymp n^{-(1-2\varepsilon)/(\gamma+2)}. \]

\[ \text{Remark 2} \] When \(\varepsilon = 0\) and in the most favorable case where \(\gamma = 1\), the rate is of order \(n^{-1/3}\).

\[ \text{Example 1} \] If \(\lambda_j = ar^j\), where \(a > 0\) and \(0 < r < 1\), by (5), we get
\[ r^{(4+2\gamma)k_n} = \frac{d}{n(1-2\varepsilon)}, d > 0, \]
which yields
\[ k_n = \left\lfloor \frac{\ln d - (1 - 2\varepsilon) \ln n}{(4 + 2\gamma) \ln r} \right\rfloor. \]

\[ \text{Example 2} \] If \(\lambda_j = aj^{-\delta}\) where \(a > 0\) and \(\delta > 1\), by (9), we get
\[ k_n = \left\lfloor e n^{(1-2\varepsilon)/(4+2\gamma)\delta} \right\rfloor, e > 0. \]
3 General Case

We consider here the empirical eigenelements of $C$ given by

$$C_n(v_{j_n}) = \lambda_{j_n} v_{j_n},$$

where $\lambda_{1n} \geq \ldots \geq \lambda_{nn} \geq 0 = \lambda_{n+1,n} = \lambda_{n+1,n} = \ldots$, and $(v_{j_n})$ constitutes an orthonormal system of $H$. We denote $\tilde{H}_{k_n}$ the space spanned by $v_{1n}, \ldots, v_{k_n,n}$. We assume in this section that each eigensubspace associated to the eigenvectors $\lambda_j$ is one dimensional. Consider the following empirical eigenvectors for identifiability reasons:

$$v'_{j_n} = \text{sgn} \langle v_{j_n}, v_j \rangle v_j, j \geq 1.$$

Consider the following estimators of $C$:

$$\tilde{C}_n = k_n \sum_{j=1}^{k_n} \lambda_{j_n} v_{j_n} \otimes v_{j_n}, \tilde{C}_{n,a} = k_n \sum_{j=1}^{k_n} \max(\lambda_{j_n}, a_n) v_{j_n} \otimes v_{j_n}.$$

Let us set

$$\tilde{\rho}_{n,a} = \tilde{\pi}^{k_n} D_n \tilde{C}_{n,a}^{-1} \pi^{k_n}$$

where $\tilde{\pi}^{k_n}$ denotes the orthogonal projector over $\tilde{H}_{k_n}$, and

$$\tilde{C}_{n,a}^{-1} = k_n \sum_{j=1}^{k_n} [\max(\lambda_{j_n}, a_n)]^{-1} v_{j_n} \otimes v_{j_n}.$$

We will show analogous results as in the previous section, using only slightly different techniques. We will use in the sequel the following numbers defined under $(H_3)$:

$$\Lambda_{k_n} = \sup_{j=1, \ldots, k_n} \frac{1}{\lambda_j - \lambda_{j+1}},$$

see [1] p. 107. Now we can give an upper bound for $E \| \tilde{\rho}_{n,a} - \rho \|_2^2$.

Lemma 2 Under $(H_1), (H_2)$ and $(H_3)$

$$E \| \tilde{\rho}_{n,a} - \rho \|_2^2 \leq \frac{c_0'}{n a_n^2} + \frac{c_1'}{n \lambda_{k_n}^2} + \frac{c_2' \Lambda_{k_n}^2}{n \lambda_{k_n}^2 a_n^2} + \frac{c_3' \Lambda_{k_n}^2}{n \lambda_{k_n}^2} + 2 \lambda_{k_n+1}^2$$

Proof. First, let us denote

$$C_{\pi^{k_n}} = k_n \sum_{j=1}^{k_n} \lambda_j v_j \otimes v_j$$

$$C_{\pi^{k_n}}^{-1} = k_n \sum_{j=1}^{k_n} \lambda_j^{-1} v_j \otimes v_j.$$

and write

$$\tilde{\rho}_{n,a} - \rho = \left( \pi^{k_n} D_n C_{\pi^{k_n}}^{-1} \pi^{k_n} - \pi^{k_n} D C_{\pi^{k_n}}^{-1} \pi^{k_n} \right) + \left( \pi^{k_n} D C_{\pi^{k_n}}^{-1} \pi^{k_n} - \pi^{k_n} D C_{\pi^{k_n}}^{-1} \pi^{k_n} \right) \pi^{k_n}$$

$$+ \left( \pi^{k_n} D C_{\pi^{k_n}}^{-1} \pi^{k_n} - \rho \right).$$
The first term may be written
\[
\tilde{z}^k \pi D_n \tilde{C}_{n,a}^{-1} \pi^k - \tilde{z}^k \pi D \tilde{C}_{\pi a}^{-1} \pi^k = \tilde{z}^k \left( D_n \tilde{C}_{n,a}^{-1} - D \tilde{C}_{\pi a}^{-1} \right) \pi^k
\]
\[
= \tilde{z}^k \left[ (D_n - D) \tilde{C}_{n,a}^{-1} + D \left( \tilde{C}_{n,a}^{-1} - C_{\pi a}^{-1} \right) \right] \pi^k
\]
\[
= \tilde{z}^k \left[ (D_n - D) \tilde{C}_{n,a}^{-1} - D \tilde{C}_{n,a}^{-1} \left( \tilde{C}_{n,a} - C_{\pi a} \right) C_{\pi a}^{-1} \right] \pi^k,
\]
hence
\[
\| \tilde{z}^k \pi D_n \tilde{C}_{n,a}^{-1} \pi^k - \tilde{z}^k \pi D \tilde{C}_{\pi a}^{-1} \pi^k \|_L^2 \\
\leq 2 \| D_n - D \|_L^2 \| \tilde{C}_{n,a}^{-1} \|_L^2 + 2 \| D \|_L^2 \| \tilde{C}_{n,a}^{-1} \|_L^2 \| C_{\pi a}^{-1} \|_L^2 \| \tilde{C}_{n,a} - C_{\pi a} \|_L^2
\]
\[
\leq 2a_n^{-2} \| D_n - D \|_L^2 + 2a_n^{-2} \| D \|_L^2 \| C_{\pi a}^{-1} \|_L^2 \| \tilde{C}_{n,a} - C_{\pi a} \|_L^2.
\]
Thus, by (7),
\[
E \| \tilde{\rho}_{n,a} - \rho \|_L^2 \leq 2 \left[ 2a_n^{-2} E \| D_n - D \|_L^2 + 2a_n^{-2} \| D \|_L^2 \| \tilde{C}_{n,a} - C_{\pi a} \|_L^2 \right] + 2E \| \tilde{z}^k \pi D \tilde{C}_{\pi a}^{-1} \pi^k - \pi^k \pi D \tilde{C}_{\pi a}^{-1} \pi^k \|_L^2 + 2E \| \pi^k \pi D \tilde{C}_{\pi a}^{-1} \pi^k - \rho \|_L^2.
\]

The third term of the right-hand side is easily bounded from above by $2\lambda_{k_n+1}^2$.

Let us now focus on the first term. \[1, \text{Th. 4.8}\] gives
\[
E \| D_n - D \|_L^2 = O \left( \frac{1}{n} \right),
\]
and clearly
\[
\| C_{\pi a}^{-1} \|_L^2 = \frac{1}{\lambda_{k_n}^2}.
\]

Moreover,
\[
E \| \tilde{C}_{n,a} - C_{\pi a} \|_L^2 \leq 2E \left( \| \tilde{C}_{n,a} - \tilde{C}_n \|_L^2 \mathbf{1}_{C_{n,a} \neq \tilde{C}_n} + \| \tilde{C}_n - C_{\pi a} \|_L^2 \mathbf{1}_{C_{n,a} \neq \tilde{C}_n} \right) + 2E \left( \| \tilde{C}_n - C_{\pi a} \|_L^2 \mathbf{1}_{C_{n,a} = \tilde{C}_n} \right)
\]
\[
+ E \left( \| \tilde{C}_{n,a} - C_{\pi a} \|_L^2 \mathbf{1}_{C_{n,a} = \tilde{C}_n} \right).
\]

Now, we find an upper bound for $P \left( \tilde{C}_{n,a} \neq \tilde{C}_n \right)$. Fortunately, the sequence $(\lambda_{jn})$ is
decreasing with respect to $j$. Therefore

$$
P(\tilde{C}_{n,a} \neq \tilde{C}_n) = P(a_n > \min_{j=1,\ldots,k_n} \lambda_{jn})$$

$$
P(\tilde{C}_{n,a} \neq \tilde{C}_n) = P(a_n > \lambda_{k_n,n})$$

$$
= P(\lambda_{k_n,n} - \lambda_{k_n} < a_n - \lambda_{k_n})$$

$$
\leq P(|\lambda_{k_n,n} - \lambda_{k_n}| \geq (1 - \beta)\lambda_{k_n})$$

$$
\leq P\left(\sup_{j=1,\ldots,k_n} |\lambda_{jn} - \lambda_j| \geq (1 - \beta)\lambda_{k_n}\right)$$

$$
\leq P\left(\|C_n - C\|_\mathcal{L} \geq (1 - \beta)\lambda_{k_n}\right)$$

$$
\leq \frac{K}{n(1 - \beta)^2\lambda_{k_n}^2}$$

with a constant $K > 0$, applying the Chebychev inequality and knowing that

$$
E\|C_n - C\|_\mathcal{L}^2 = O\left(\frac{1}{n}\right).$$

Note that

$$\|\tilde{C}_{n,a} - \tilde{C}_n\|_\mathcal{L} \leq a_n^2$$

and that

$$E\left\|\tilde{C}_n - C_n\|_\mathcal{L}^2 = E\left\|\sum_{j=1}^{k_n} \lambda_{jn} v_{jn} \otimes v_{jn} - \sum_{j=1}^{k_n} \lambda_j v_j \otimes v_j\right\|_\mathcal{L}^2$$

$$\leq 2E\left\|\sum_{j=1}^{k_n} \lambda_{jn} v_{jn} \otimes v_{jn} - \sum_{j=1}^{k_n} \lambda_j v_j \otimes v_j\right\|_\mathcal{L}^2 + 2E\left\|\sum_{j=1}^{k_n} \lambda_j v_j \otimes v_j - \sum_{j=1}^{k_n} \lambda_j v_j \otimes v_j\right\|_\mathcal{L}^2$$

$$\leq 2E\left\|\sum_{j=1}^{k_n} (\lambda_{jn} - \lambda_j) v_{jn} \otimes v_{jn}\right\|_\mathcal{L}^2 + 2E\left\|\sum_{j=1}^{k_n} \lambda_j (v_{jn} \otimes v_{jn} - v_j \otimes v_j)\right\|_\mathcal{L}^2.$$

But

$$E\left\|\sum_{j=1}^{k_n} (\lambda_{jn} - \lambda_j) v_{jn} \otimes v_{jn}\right\|_\mathcal{L}^2 \leq E\sup_{j \geq 1} |\lambda_{jn} - \lambda_j|^2$$

$$\leq E\|C_n - C\|_\mathcal{L}^2 = O\left(\frac{1}{n}\right),$$

by [1, Th. 4.1], and

$$v_{jn} \otimes v_{jn} - v_j \otimes v_j = v_{jn} \otimes v_{jn} - v'_{jn} \otimes v'_{jn}$$

$$= (v_{jn} - v'_{jn}) \otimes v_{jn} + v'_{jn} \otimes (v_{jn} - v'_{jn}),$$

9
Theorem 2 Suppose that \((H_1), (H_2)\) and \((H_3)\) hold, and that there exist \(\alpha > 0, 0 < \beta < 1, \varepsilon < 1/2\) and \(\gamma \geq 1\) such that

\[
\lambda_{k_n}^\gamma \leq a_n \leq \beta \lambda_{k_n},
\]

so

\[
E \left\| \sum_{j=1}^{k_n} \lambda_j (v_{jn} \otimes v_{jn} - v_j \otimes v_j) \right\|_L^2 \leq 2E \left\| \sum_{j=1}^{k_n} \lambda_j (v_{jn} - v_j') \otimes v_{jn} \right\|_L^2 + 2E \left\| \sum_{j=1}^{k_n} \lambda_j v_j' \otimes (v_{jn} - v_j') \right\|_L^2 \leq 4 \sup_{j=1, \ldots, k_n} |\lambda_j|^2 E \left\| v_{jn} - v_j' \right\|_L^2 \leq 4 |\lambda_1|^2 E \sup_{j=1, \ldots, k_n} \left\| v_{jn} - v_j' \right\|_L^2 \]

by [1, Lemma 4.3]. Accordingly by [1, Th. 4.1],

\[
E \left\| \sum_{j=1}^{k_n} \lambda_j (v_{jn} \otimes v_{jn} - v_j \otimes v_j) \right\|_L^2 = O \left( \frac{\Lambda_{k_n}^2}{n} \right).
\]

For the second term of \((8)\), we write

\[
E \left\| \tilde{\pi}^{k_n} DC^{-1}_{\pi} \tilde{\pi}^{k_n} - \pi^{k_n} DC^{-1}_{\pi} \pi^{k_n} \right\|_L^2 \leq 2E \left\| \tilde{\pi}^{k_n} DC^{-1}_{\pi} \pi^{k_n} - \pi^{k_n} DC^{-1}_{\pi} \pi^{k_n} \right\|_L^2 + 2E \left\| \tilde{\pi}^{k_n} DC^{-1}_{\pi} \pi^{k_n} \right\|_L^2 \leq 2E \left\| \tilde{\pi}^{k_n} DC^{-1}_{\pi} \pi^{k_n} \right\|_L^2 \leq \frac{L}{\lambda_{k_n}^2} E \left\| \tilde{\pi}^{k_n} - \pi^{k_n} \right\|_L^2 \]

with a constant \(L > 0\). Finally, notice that by a similar calculus as previously,

\[
E \left\| \tilde{\pi}^{k_n} - \pi^{k_n} \right\|_L^2 = E \left\| \sum_{j=1}^{k_n} v_{jn} \otimes v_{jn} - v_j' \otimes v_j' \right\|_L^2 = O \left( \frac{\Lambda_{k_n}^2}{n} \right).
\]

Consequently, \((8)\) entails

\[
E \| \tilde{\rho}_{n,a} - \rho \|_L^2 \leq \frac{c_0'}{na_n^2} + \frac{c_1'}{n \lambda_{k_n}^4} + \frac{c_2' \Lambda_{k_n}^2}{n a_n^2 \lambda_{k_n}^2} + \frac{c_3' \Lambda_{k_n}^2}{n \lambda_{k_n}^2} + 2\lambda_{k_n}^2 + 1
\]

with positive constants \(c_i'\). ■
then
\[ E \| \hat{\rho}_{n,a} - \rho \|_L^2 = O \left( \frac{\lambda_{kn}^2}{n(1-2\varepsilon)\lambda_{kn}^{2(1+\gamma)}} \right) + O \left( \lambda_{kn}^2 \right) \]

**Proof.** It is an easy consequence of (6), using the inequalities \( \alpha \lambda_{kn} \leq a_n \) and \( \lambda_{kn+1} \leq \lambda_{kn} \). ■

**Remark 3** The optimal choice of \( \lambda_{kn} \) is such that:
\[ \lambda_{kn}^2 = \frac{c' \Lambda_{kn}^2}{n(1-2\varepsilon)} \lambda_{kn}^{2+2\gamma}, \text{ i.e. } \lambda_{kn}^{4+2\gamma} = \frac{c' \Lambda_{kn}^2}{n(1-2\varepsilon)}, c' > 0. \] (9)

The rate of convergence in quadratic mean is then of order
\[ \lambda_{kn}^2 \asymp \left( \frac{\Lambda_{kn}^2}{n(1-2\varepsilon)} \right)^{1/(2+\gamma)}. \]

**Example 3** If \( \lambda_j = ar^j \), where \( a > 0 \) and \( 0 < r < 1 \), by (9), we get
\[ r^{(6+2\gamma)kn} = \frac{d'}{n(1-2\varepsilon)}, d' > 0 \]
which yields
\[ k_n = \left\lfloor \frac{\ln d' - (1-2\varepsilon) \ln n}{(6+2\gamma) \ln r} \right\rfloor. \]
The rate of convergence in quadratic mean is then of order
\[ r^{-2(1-2\varepsilon)\ln n/(6+2\gamma)\ln r} = n^{-(1-2\varepsilon)/(\gamma+3)}. \]

**Example 4** If \( \lambda_j = aj^{-\delta} \), where \( a > 0 \) and \( \delta > 1 \), a few calculations yield
\[ \Lambda_{kn}^2 \approx Mk_n^{2(\delta+1)}, M \geq 0, \]
and by (9), we get
\[ k_n = \left\lfloor e' n^{(1-2\varepsilon)/[2\delta(\gamma+3)+2]} \right\rfloor, e' > 0. \]
The rate of convergence in quadratic mean is then of order \( k_n^{-2\delta} \), i.e.
\[ n^{-\delta(1-2\varepsilon)/[\delta(\gamma+3)+1]}. \]

**Remark 4** When \( \varepsilon = 0 \) and in the most favorable case where \( \gamma = 1 \), the rate of convergence in example 4 is of order \( n^{-\delta/(4\delta+1)} \) and therefore asymptotically of order \( n^{-1/4} \) as \( \delta \to \infty \), which is the rate of convergence in example 3.
References


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