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Serge GUILLAS

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Institut de Statistique de l’Université de Paris
Laboratoire de Statistique Théorique et Appliquée

Université Paris VI
Tour 45-55, Boîte 158
4, place Jussieu
75252 Paris Cedex 05
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Serge GUILLAS

Université Paris VI, ISUP-LSTA
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S. Guillas
Université Paris 6
Ecole des Mines de Douai
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Abstract

In this paper, we consider a Hilbert space valued autoregressive stochastic sequence \((X_n)\) with several regimes. We suppose that the underlying process \((I_n)\) which drives the evolution of \((X_n)\) is stationary. Under some dependence assumptions on \((I_n)\), we prove the existence of a unique stationary solution, and with a symmetric compact autocorrelation operator, we can state a Law of Large Numbers with rates and the consistency of the covariance estimator. For the cross-covariance operators, the boundedness of \((X_n)\) ensures the same type of consistency. An overall hypothesis states that the regimes where the autocorrelation operator’s norm is greater than one should be rarely visited.

1 Introduction

Autoregressive Hilbertian processes of order one (ARH(1)) have been extensively investigated by Bosq \[4, 5\], mainly because of the interest for prediction of continuous time stochastic processes (e.g. electricity consumption \[8\], traffic \[2\], or climatic variation \[3\]). The inclusion of exogenous variables in an additive way may improve the model. Therefore, it has recently been considered theoretically \[11\] and applied to pollution forecasting \[9\]. However, in the case of real-valued time series, several authors also examined the non-additive inclusion. For instance, we can mention regime models (according to Tjostheim’s terminology \[21\]) of the form

\[X_t = \theta_t X_{t-1} + e_t\] (1)

where \((e_t)\) is a noise and \((\theta_t)\) is a stochastic process. When \(\theta_t\) is regulated by \(X_{t-1}\), \[1\] results in ”(Self-Exciting) Threshold Autoregressive processes” (SETAR or possibly TAR). On the other hand, if \((\theta_t)\) is a Markov chain having a finite state space, we say that the model is doubly stochastic \[20\]. ”Random Coefficient Autoregressive Models” (RCA) \[16\] appear in the particular case where \((\theta_t)\) is i.i.d.

Tong \[22\], \[23\] studied TAR processes and an application to the prediction of ozone can be found in \[15\]. These nonlinear processes should fit better to phenomena with sudden changes such as river flows.

Petrucelli and Woolford \[17\] considered the real valued model

\[Z_t = \phi_1 Z_{t-1}^+ + \phi_2 Z_{t-1}^- + \varepsilon_t, t = 1, 2, \ldots\]
where $x^+ = \max(x,0), x^- = \min(x,0)$ and $(\varepsilon_t)$ is a white noise. They showed that $(Z_t)$ is ergodic if and only if

$$\phi_1 < 1, \phi_2 < 1, \phi_1 \phi_2 < 1,$$

which is less restrictive than

$$|\phi_1| < 1, |\phi_2| < 1.$$

Similarly, Brandt [7] and Bougerol and Picard [9] made use of the negativity of the top Lyapunov exponent in order to ensure the existence of a stationary solution, but Yao [24] pointed out that this criterion is difficult to apply, proposing to look at the spectral radius of the autocorrelation matrix in a multivariate framework. See also Pourahmadi [13] in this context, Franck and Zakoïan [10] for extensions to the multivariate ARMA case, Yao and Attali [25] for an investigation of the nonlinear case, and Horst [14] for non-stationary $(\theta_t)$ and $(\epsilon_t)$.

From our ARH(1) point of view, the model imposes conditions of the following type (see [5]):

$$\exists j, \|\rho_j\|_L < 1,$$

for the existence of a stationary solution to the equation

$$X_n = \rho(X_{n-1}) + \varepsilon_n,$$

where $H$ is a real and separable Hilbert space - usually an infinite-dimensional one - with norm $\|\cdot\|$, $\rho$ is a bounded linear operator on $H$, $\|\cdot\|_L$ is the linear norm of linear operators on $H$, and $(\varepsilon_n)$ is a strong Hilbertian white noise (SHWN), that is a sequence of i.i.d. random variables with values in $H$ satisfying

$$\forall n \in \mathbb{Z}, E\varepsilon_n = 0, 0 < E\|\varepsilon_n\|^2 = \sigma^2 < \infty.$$

Let $(I_n)_{n \in \mathbb{Z}}$ be a strictly stationary sequence of random variables taking their values in $\{0, 1\}$, and $(\varepsilon_n)_{n \in \mathbb{Z}}$ be a SHWN. Let $\rho_0$ and $\rho_1$ be two bounded linear operators on $H$. We consider the following model:

$$\begin{cases}
X_n = \rho_0(X_{n-1}) + \varepsilon_n & \text{if } I_n = 0 \\
X_n = \rho_1(X_{n-1}) + \varepsilon_n & \text{if } I_n = 1
\end{cases} \quad (2)$$

which can be also written down as

$$X_n = \rho_{I_n}(X_{n-1}) + \varepsilon_n. \quad (3)$$

In this paper, we extend doubly stochastic models to Hilbert space valued processes. When one of the two choices leads to an explosive case, for instance if $\|\rho_j\| \geq 1$ for all $j$, the alternative choice may stabilize the behavior as we will see hereafter. Note that with this kind of sample paths, it seemed reasonable to study extremes for such processes (see de Haan et al. [13] in the particular case of RCA models). In section 2, we establish the existence and uniqueness of a stationary solution under the assumption that $(I_n)_{n \in \mathbb{Z}}$ is $(m-1)$-dependent, with a positive integer $m$. Then, in section 3 we state a Law of Large Numbers with rates, under the assumption that $\rho_{I_n}$ is symmetric and compact and $(I_n)_{n \in \mathbb{Z}}$ is i.i.d. (RCA models). Finally, in section 4 we show the consistency of some estimators of the covariance and cross-covariance operators. Note that our paper is devoted to the two regimes case for simplicity, but it is easy to generalize to the multiple regimes case.
2 Stationary solution

We start with two technical Lemmas. First, let us recall the following result of Bass [1] (see also [19]).

**Lemma 2.1 (Bass 1955)** Let $Y_1, ..., Y_n$ be non-negative random variables. Then,

$$E(Y_1...Y_n) \leq \int_0^1 Q_{Y_1}(u)...Q_{Y_n}(u)du$$

where $Q_X$ denotes the quantile function of the r.v. $|X|$ defined by

$$Q_X(u) = \inf \{t, P(|X| > t) \leq u\}.$$ 

We then infer the following upper bound from Bass Lemma.

**Lemma 2.2** Let $a$ and $b$ be two positive real numbers. Let $Y_1, ..., Y_n$ be identically distributed random variables such that

$$P(Y_1 = a) = 1 - q, P(Y_1 = b) = q.$$ 

Then

$$E(Y_1...Y_n) \leq (1 - q) a^n + qb^n.$$  \hspace{1cm} (4)

**Proof.** Apply Lemma 2.1

$$E[Y_1...Y_n] \leq \int_0^1 (Q_Y(u))^n du,$$

with $Q_Y(u) := Q_{Y_1}(u)$. If $b \leq a$ then

$$P(Y_1 > t) = \begin{cases} 
0 & \text{if } t \geq a \\
1 - q & \text{if } t \in [b,a] \\
1 & \text{otherwise}
\end{cases}$$

and if $a \leq b$ then

$$P(Y_1 > t) = \begin{cases} 
0 & \text{if } t \geq b \\
q & \text{if } t \in [a,b] \\
1 & \text{otherwise}\end{cases}. $$

Hence, in the case where $b \leq a$,

$$Q_Y(u) = \begin{cases} 
a & \text{if } 0 \leq u < 1 - q \\
b & \text{if } 1 - q \leq u < 1
\end{cases}$$

and in the case where $a \leq b$,

$$Q_Y(u) = \begin{cases} 
b & \text{if } 0 \leq u < q \\
a & \text{if } q \leq u < 1
\end{cases}.$$ 

Thus, in general,

$$\int_0^1 (Q_Y(u))^n du = (1 - q) a^n + qb^n.$$
Theorem 2.1 Suppose that \((I_n)\) is \((m - 1)\)-dependent and that
\[
m_4 := E \|\varepsilon_0\|^4 < \infty.
\]
Set
\[
q = P(I_n = 1),
\]
and
\[
c_m = (1 - q) \|\rho_0\|^{8m} + q \|\rho_1\|^{8m}.
\]
If \(c_m < 1\), then (2) has a stationary solution given by
\[
X_n = \sum_{j=0}^{\infty} \left( \prod_{p=0}^{j-1} \rho_{I_n-p} \right) (\varepsilon_{n-j}),
\]
the series converges in \(L^2_H(P)\) (that is in the mean square sense), with the convention
\[
\prod_{p=0}^{j-1} \rho_{I_n-p} = \Id, \text{ pour } j = 0.
\]
Under the condition
\[
E \|X_0\|^4 < \infty,
\]
this solution is unique.

Proof. For the existence:
\[
\Delta_l' := \left\| \sum_{j=l}^{\nu} \left( \prod_{p=0}^{j-1} \rho_{I_n-p} \right) (\varepsilon_{n-j}) \right\|^2_{L^2_H(P)} = \sum_{j=l}^{\nu} \left\| \left( \prod_{p=0}^{j-1} \rho_{I_n-p} \right) (\varepsilon_{n-j}) \right\|^2_{L^2_H(P)}
\]
since the \(\varepsilon_n\) are orthogonals. Therefore,
\[
\Delta_l' \leq \sum_{j=l}^{\nu} E \left( \left\| \prod_{p=0}^{j-1} \rho_{I_n-p} \right\|^2 \left\| \varepsilon_{n-j} \right\|^2_H \right) \leq \sum_{j=l}^{\nu} \left( E \left[ \prod_{p=0}^{j-1} \|\rho_{I_n-p}\|_H^4 \right] \right)^{1/2} \left( E \left[ \|\varepsilon_{n-j}\|^4_H \right] \right)^{1/2} \left( m_j \right)^{1/2}
\]
It remains to find an upper bound to \( E \left[ \prod_{p=0}^{j-1} \|\rho_{I_n-p}\|_H^4 \right] \). For that purpose, remark that the r.v. \(Y_p = \|\rho_{I_n-p}\|_H^4\) are identically distributed, and use the \((m - 1)\)-dependence of the sequence \((I_n)\) and then of the sequence \((Y_n)\). We may now write down:
\[
Y_1...Y_j = Y_1...Y_m Y_{m+1}...Y_{2m} Y_{2m+1}...Y_{km} Y_{km+1}...Y_{km+r}
\]
where \( j = km + r \) is the expression of the euclidean division of \( j \) by \( m \) (\( 0 \leq r < m \)).

Let us denote

\[
U_1 = Y_1 ... Y_m \\
V_1 = Y_m Y_{m+1} ... Y_{2m} \\
U_2 = Y_{2m+1} ... Y_{3m} \\
\vdots
\]

so that

\[
Y_1 ... Y_m Y_{m+1} ... Y_{2m} Y_{2m+1} ... Y_{km} Y_{km+1} ... Y_{km+r} = \begin{cases} 
U_1 V_1 U_2 ... U_{\lfloor k/2 \rfloor} Y_{km+1} ... Y_{km+r} & \text{if } k \text{ odd} \\
U_1 V_1 U_2 ... V_{\lfloor k/2 \rfloor} Y_{km+1} ... Y_{km+r} & \text{if } k \text{ even}
\end{cases}
\]

Hence, for odd \( k \), and denoting \( V_{\lfloor k/2 \rfloor + 1} = Y_{km+1} ... Y_{km+r} \)

\[
E \left[ \prod_{p=0}^{j-1} Y_p \right] \leq \left( E \left[ \prod_{i=0}^{\lfloor k/2 \rfloor + 1} U_i^2 \right] \right)^{1/2} \left( E \left[ \prod_{i=0}^{\lfloor k/2 \rfloor + 1} V_i^2 \right] \right)^{1/2} \\
\leq \left[ \left( 1 - q \right) \left\| \rho_0 \right\|^8 + q \left\| \rho_1 \right\|^8 \right]^{\lfloor k/2 \rfloor + 1} \left[ \left( 1 - q \right) \left\| \rho_0 \right\|^8 + q \left\| \rho_1 \right\|^8 \right]^{\lfloor k/2 \rfloor}^{1/2}
\]

\[
\left( 1 - q \right) \left\| \rho_0 \right\|^8 + q \left\| \rho_1 \right\|^8
\]

by Lemma 2.2. Accordingly, for odd \( k \),

\[
E \left[ \prod_{p=0}^{j-1} Y_p \right] \leq \left[ \left( 1 - q \right) \left\| \rho_0 \right\|^8 + q \left\| \rho_1 \right\|^8 \right]^{1/2} \left[ \left( 1 - q \right) \left\| \rho_0 \right\|^8 + q \left\| \rho_1 \right\|^8 \right]^{\lfloor k/2 \rfloor}
\]

upper bound which holds also for even \( k \)’s by a similar argument. If we set

\[
M_m = \sup_{r=0,...,m-1} \left[ \left( 1 - q \right) \left\| \rho_0 \right\|^8 + q \left\| \rho_1 \right\|^8 \right]^{1/2}
\]

and

\[
c_m = \left( 1 - q \right) \left\| \rho_0 \right\|^8 + q \left\| \rho_1 \right\|^8
\]

we obtain

\[
E \left[ \prod_{p=0}^{j-1} Y_p \right] \leq M_m c_m^{\lfloor k/2 \rfloor}
\]

and

\[
\left( E \left[ \prod_{p=0}^{j-1} \left\| \rho_{I_{n-r}} \right\|_E \right] \right)^{1/2} \leq \left( M_m c_m^{\lfloor k/2 \rfloor} \right)^{1/2}.
\]

And since

\[
j = km + r \\
k = \frac{j - r}{m} > \frac{j - m}{m} = \frac{j}{m} - 1,
\]

5
The term $L^a$ convergent series. Thus $\Delta l$ and $\rho$ since $K > 0$, with $a > 0$ we get

\[
\sum_{j=0}^{\infty} \left( \prod_{p=0}^{j-1} \rho a_{n-p} \right) (\varepsilon_{n-j}).
\]

Since $\rho_0$ and $\rho_1$ are bounded,

\[
W_n - \rho_1^n (W_{n-1}) = \sum_{j=0}^{\infty} \left( \prod_{p=0}^{j-1} \rho a_{n-p} \right) (\varepsilon_{n-j}) - \sum_{j=0}^{\infty} \rho_1^n \left( \prod_{p=0}^{j-1} \rho a_{n-1-p} \right) (\varepsilon_{n-1-j})
\]

\[
= \sum_{j=0}^{\infty} \left( \prod_{p=0}^{j-1} \rho a_{n-p} \right) (\varepsilon_{n-j}) - \sum_{j=0}^{\infty} \left( \prod_{p=0}^{j} \rho a_{n-p} \right) (\varepsilon_{n-1-j})
\]

\[
= \sum_{j=0}^{\infty} \left( \prod_{p=0}^{j-1} \rho a_{n-p} \right) (\varepsilon_{n-j}) - \sum_{j'=1}^{\infty} \left( \prod_{p=0}^{j'-1} \rho a_{n-p} \right) (\varepsilon_{n-j'})
\]

\[
= \varepsilon_n.
\]

Conversely, let $(X_n)$ be a stationary solution of (2) satisfying

\[
E \|X_0\|^4 < \infty,
\]

then by a straightforward induction, it follows

\[
X_n = \left( \prod_{p=0}^{k} \rho a_{n-p} \right) (X_{n-k-1}) + \sum_{j=0}^{k} \left( \prod_{p=0}^{j-1} \rho a_{n-p} \right) (\varepsilon_{n-j})
\]

(8)

and

\[
E \left\| X_n - \sum_{j=0}^{k} \left( \prod_{p=0}^{j-1} \rho a_{n-p} \right) (\varepsilon_{n-j}) \right\|^2 \leq E \left( \prod_{p=0}^{k} \rho a_{n-p} \right) (X_{n-k-1})^2 \leq \left( E \left\| \prod_{p=0}^{k} \rho a_{n-p} \right\|_{\mathcal{L}}^4 \right)^{1/2} \left( E \|X_{n-k-1}\|^4 \right)^{1/2}.
\]

The term $\left( E \|X_{n-k-1}\|^4 \right)^{1/2}$ is constant because of the stationarity of $(X_n)$. And the use of the bound in (7) entails the conclusion:

\[
E \left\| X_n - \sum_{j=0}^{k} \left( \prod_{p=0}^{j-1} \rho a_{n-p} \right) (\varepsilon_{n-j}) \right\|^2 \to 0
\]
when \( k \to \infty \). □

3 Law of large numbers

In the sequel, we will assume that \( \rho \) is symmetric and compact of the form

\[
\rho = \sum_{j=0}^{\infty} A_j \otimes e_j \otimes e_j
\]

where \((e_j)\) is a fixed orthonormal basis of \( H \), and \((A_{j,0}), (A_{j,1})\) are two non-increasing, non-negative sequences tending to 0, such that for all \( j \) and all \( n \)

\[
\rho(t_n) = A_{j,t_n} e_j.
\]

Actually, only the eigenvalues are random, the change of regime is just in terms of level of influence.

The following Proposition will be a technical tool used in the proof of our results. It generalizes the property - holding in the real-valued context - that it is possible to take out of the conditional expectation random variables which are measurable with respect to the considered \( \sigma \)-algebra.

**Proposition 3.1** Let \((L_j)_{j \in \mathbb{N}}\) be a sequence of real random variables defined on a probability space \((\Omega, \mathcal{A}, P)\) such that

\[
L = \sum_{j=0}^{\infty} L_j e_j \otimes e_j
\]

is a bounded linear operator on \( H \), \((e_j)\) being a fixed orthonormal basis of \( H \). Let \( \mathcal{A}_0 \) be a sub-\( \sigma \)-algebra of \( \mathcal{A} \). Let \( X \) be an integrable \( H \)-valued random variable. If the sequence \((L_j)_{j \in \mathbb{N}}\) is \( \mathcal{A}_0 \)-measurable, then

\[
E^{\mathcal{A}_0}[L(X)] = L(E^{\mathcal{A}_0}[X]).
\]

**Proof.** In order to show that \( L(E^{\mathcal{A}_0}[X]) \) is the conditional expectation - see \[5 \] (1.34) for the definition - of \( L(X) \) relative to \( \mathcal{A}_0 \), it is sufficient to show that for all \( \mathcal{A}_0 \)-mesurable \( A \)

\[
E[1_A L(E^{\mathcal{A}_0}[X])] = E[1_A L(X)]
\]

i.e.

\[
E \left[ \left( 1_A L \left( E^{\mathcal{A}_0} [X] \right), e_j \right) \right] = E \left[ (1_A L(X), e_j) \right], \quad j \in \mathbb{N}
\]

i.e.

\[
E \left[ 1_A L_j \left( E^{\mathcal{A}_0} [X], e_j \right) \right] = E \left[ 1_A L_j \langle X, e_j \rangle \right], \quad j \in \mathbb{N}
\]

i.e.

\[
E \left[ 1_A L_j E^{\mathcal{A}_0} \langle X, e_j \rangle \right] = E \left[ 1_A L_j \langle X, e_j \rangle \right], \quad j \in \mathbb{N}.
\]

But, since the random variables \( 1_A L_j \) and \( \langle X, e_j \rangle \) are \( \mathbb{R} \)-valued, we recognize here the well known result about conditional expectation:

\[
E^{\mathcal{A}_0}[1_A L_j \langle X, e_j \rangle] = 1_A L_j E^{\mathcal{A}_0} \langle X, e_j \rangle.
\]

□
The Law of Large Numbers will be stated under the hypothesis:

\[(I_n)\ \text{i.i.d.}\]  \hspace{1cm} (H1)

Note that under \((H1)\), \((I_n)\) is \((m-1)\)-dependent with \(m = 1\). Such an influence on the change of regime connects our model with RCA ones.

We will also assume the following hypothesis:

\[(I_n)\ \text{and } (\varepsilon_n) \text{ are independent.}\]  \hspace{1cm} (H2)

This assumption tells us that the two influences (the index of the operator and the additive noise) are not linked: we may talk of a double noise.

Set \(S_n = X_1 + ... + X_n\).

**Theorem 3.1 (LGN)** Under \((H1)\) and \((H2)\), under all the assumptions of Theorem 2.1, and if

\[(1-q) \|\rho_0\| + q \|\rho_1\| < 1,\]

then

\[E \left\| \frac{S_n}{n} \right\|^2 = O\left( \frac{1}{n} \right)\]  \hspace{1cm} (9)

and

\[\frac{n^{1/4} S_n}{(\ln n)^{3/2}} \rightarrow 0 \ a.s., \beta > 1/2.\]  \hspace{1cm} (10)

**Proof.** Note that

\[E \left\| X_n + ... + X_{n+p-1} \right\|^2 = \sum_{n \leq j, j' \leq n+p-1} E \langle X_j, X_{j'} \rangle.\]

by stationarity of \((X_n)\). We use now \((8)\) with \(n = h\) and \(k = h - 1\) to compute \(\langle X_0, X_h \rangle\):

\[\langle X_0, X_h \rangle = \left\langle X_0, \sum_{j=0}^{h-1} \left( \prod_{p=0}^{j-1} \rho_{h-p} \right) \langle \varepsilon_{h-j} \rangle \right\rangle + \langle X_0, \rho_{h-1} ... \rho_1 (X_0) \rangle.\]  \hspace{1cm} (11)

By another way

\[E \left\langle X_0, \sum_{j=0}^{h-1} \left( \prod_{p=0}^{j-1} \rho_{h-p} \right) \langle \varepsilon_{h-j} \rangle \right\rangle = \sum_{j=0}^{h-1} E \left[ \left\langle X_0, \left( \prod_{p=0}^{j-1} \rho_{h-p} \right) \langle \varepsilon_{h-j} \rangle \right\rangle \right] \]

\[= \sum_{j=0}^{h-1} \left[ \sum_i \langle X_0, e_i \rangle e_i \sum_i A_{i, h} ... A_{i, h-j-1} \langle \varepsilon_{h-j}, e_i \rangle e_i \right].\]

But

\[\sum_i E \left[ |A_{i, h} ... A_{i, h-j-1} \langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle| \right] < \infty.\]
Indeed, supposing without loss of generality that \( \| \rho_1 \| \geq \| \rho_0 \| \),

\[
\sum_i E \left[ \langle A_i, I_i \rangle \langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle \right] \leq \| \rho_1 \|^2 \sum_i E \left[ \| \langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle \| \right]
\]

\[
\leq \| \rho_1 \|^2 \left[ \frac{1}{2} \sum_i |\langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle|^2 + |\langle \varepsilon_{h-j}, e_i \rangle|^2 \right]
\]

\[
\leq \| \rho_1 \|^2 \left( \frac{1}{2} E \| X_0 \|^2 + E \| \varepsilon_0 \|^2 \right) \leq \infty.
\]

Consequently,

\[
E \left[ X_0, \sum_{j=0}^{h-1} \left( \prod_{p=0}^{j-1} \rho_{I_{h-p}} \right) \langle \varepsilon_{h-j} \rangle \right] = \sum_{j=0}^{h-1} \sum_i E \left[ A_i, I_i \langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle \right] \]

\[
= \sum_{j=0}^{h-1} \sum_i E \left[ A_i, I_i \langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle \right] \sigma (I_1, \ldots, I_h)]
\]

\[
= \sum_{j=0}^{h-1} \sum_i E \left[ A_i, I_i \langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle \right] \sigma (I_1, \ldots, I_h)]
\]

Now, owing to the independence of \( X_0 \) and \( \varepsilon_{h-j} \) with \( \sigma (I_1, \ldots, I_h) \),

\[
E \left[ \langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle \sigma (I_1, \ldots, I_h) \right] = E \left[ \langle X_0, e_i \rangle \langle \varepsilon_{h-j}, e_i \rangle \right] = 0.
\]

and therefore

\[
E \left[ X_0, \sum_{j=0}^{h-1} \left( \prod_{p=0}^{j-1} \rho_{I_{h-p}} \right) \langle \varepsilon_{h-j} \rangle \right] = 0.
\]

We manage the second term of (11) in a similar way

\[
E \left[ X_0, \rho_{I_h} \cdots \rho_{I_1} \langle X_0 \rangle \right] = E \left[ \left( \sum_i \langle X_0, e_i \rangle e_i, \sum_i A_i, I_i \langle X_0, e_i \rangle e_i \right) \right]
\]

\[
= E \left[ \sum_i A_i, I_i \langle X_0, e_i \rangle^2 \right]
\]

\[
= \sum_i E \left[ A_i, I_i \langle X_0, e_i \rangle^2 \right]
\]

because all the terms are non negative, so

\[
E \left[ X_0, \rho_{I_h} \cdots \rho_{I_1} \langle X_0 \rangle \right] = \sum_i E \left[ A_i, I_i \langle X_0, e_i \rangle^2 \right] \sigma (I_1, \ldots, I_h)]
\]

\[
= \sum_i E \left[ A_i, I_i \langle X_0, e_i \rangle^2 \right] \sigma (I_1, \ldots, I_h)]
\]

But, owing to the independence of \( X_0 \) with \( I_1, \ldots, I_h \),

\[
E \left[ \langle X_0, e_i \rangle^2 \right] \sigma (I_1, \ldots, I_h)] = E \left[ \langle X_0, e_i \rangle^2 \right].
\]
Hence,
\[ E\langle X_0, \rho_{I_h}...\rho_{I_t}(X_0) \rangle = \sum_i E[A_i, I_h...A_i, I_t] E\langle X_0, e_i \rangle^2. \]

Since
\[ E[A_i, I_h...A_i, I_t] \leq E[A_1, I_h...A_1, I_t] \]
\[ \leq \sum_{j=0}^h C_j q^j (1 - q)^{h-j} \|\rho_1\|^j \|\rho_0\|^h-j \]
\[ = ((1 - q) \|\rho_0\| + q \|\rho_1\|)^h, \]
we get
\[ E\langle X_0, \rho_{I_h}...\rho_{I_t}(X_0) \rangle \leq ((1 - q) \|\rho_0\| + q \|\rho_1\|)^h. \]

Accordingly,
\[ E\|X_n + ... + X_{n+p-1}\|^2 \leq 2p E\|X_0\|^2 \sum_{h=0}^{p-1} ((1 - q) \|\rho_0\| + q \|\rho_1\|)^h. \]

As \((1 - q) \|\rho_0\| + q \|\rho_1\| < 1, \)
\[ E\|X_n + ... + X_{n+p-1}\|^2 = O(p), \]
which entails \([9], \) and applying \([5, Cor. 2.3], \) we obtain \([10]. \)

4 Estimation of the covariance estimator

At this point, we use the same method as in \([4]. \) Let us recall that the covariance operator and the empirical covariance operator are defined respectively by
\[ C(.) = E[(X_0, .) X_0] \]
\[ C_n(.) = \frac{1}{n} \sum_{t=1}^n (X_t, .) X_t. \]

**Theorem 4.1** Under \([H1] \) and \([H2], \) under all the assumptions of Theorem 2.1 and if
\[ \alpha_2 := (1 - q) \|\rho_0\|^2 + q \|\rho_1\|^2 < 1, \]
then
\[ E\|C_n - C\|_S^2 = O\left(\frac{1}{n}\right), \quad (12) \]
\[ \|C_n - C\|_S \xrightarrow{n \to \infty} 0 \text{ a.s.} \quad (13) \]
Let us find an upper bound for $E$ or $p$ terms of the following form vanish for $Z$ Schmit operators on $E$ owing to independence properties. But because the random variables $\rho$ by $\epsilon_1 + \ldots + \rho_{11} - \rho_{12}$ + $\rho_{13} - \rho_{14}$ $(X_0)$ owing to relation $8$. We only find in the first sum of $14$ terms of the form

$$E \left( \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \right).$$

or

$$E \left( \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \right).$$

Terms of the following form vanish for $p \neq k$ : as a matter of fact,

$$E \left( \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \right) = E \left( \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \langle X_0, e_j \rangle \right)$$

owing to independence properties. But

$$E \left( \langle \rho_{1p} \ldots \rho_{1p} \langle \epsilon_{p-1}, \epsilon_j \rangle \langle \epsilon_{k-1}, \epsilon_j \rangle \right) = E \left[ E \left( \langle \rho_{1p} \ldots \rho_{1p} \langle \epsilon_{p-1}, \epsilon_j \rangle \langle \epsilon_{k-1}, \epsilon_j \rangle \right) \right]$$

because the random variables $\rho_{1p} \ldots \rho_{1p} \langle \epsilon_{p-1}, \epsilon_j \rangle$ and $\rho_{1p} \ldots \rho_{1p} \langle \epsilon_{k-1}, \epsilon_j \rangle$ are conditionally independent with respect to $(I_n)$ by the independence of $\epsilon_{p-1}$ and $\epsilon_{k-1}$. Now

$$E \left( \langle \rho_{1p} \ldots \rho_{1p} \langle \epsilon_{k-1}, \epsilon_j \rangle \right) = E \left[ E \left( \langle \rho_{1p} \ldots \rho_{1p} \langle \epsilon_{k-1}, \epsilon_j \rangle \right) \right]$$

$$= E \left( \langle \rho_{1p} \ldots \rho_{1p} \langle \epsilon_{k-1}, \epsilon_j \rangle \right)$$

$$= E \left( \langle \rho_{1p} \ldots \rho_{1p} E \left( \epsilon_{k-1} \right) \right).$$

**Proof.** Define the following random variables with values in the space $S$ of Hilbert-Schmidt operators on $H$:

$$Z_t = \langle X_t, . \rangle X_t - C, \ t = 1, 2, \ldots$$

Note that $(Z_t)$ is a stationary process. Moreover,

$$E \|Z_t\|^2 \leq 2E \|\langle X_t, . \rangle X_t\|^2 + 2\|C\|^2$$

$$\leq 2E \|X_0\|^4 + 2\|C\|^2,$$

by [5] (1.49), (1.55)]. Hence,

$$E \|Z_t\|^2 \leq 2E \|X_0\|^4 + 2 \left( E \|X_0\|^2 \right)^2 = 2M < \infty$$

by [5] (1.59)].

Because of the stationarity,

$$E \|Z_n + \ldots + Z_{n+p-1}\|^2 \leq 2pM + 2 \sum_{h=1}^{p-1} (p - h) E \langle Z_0, Z_h \rangle_S.$$
by Proposition 3.1. Owing to independence properties and knowing that \( \varepsilon_0 \) is centered,
\[
E \left[ \varepsilon_{k-1} \right] = E[\varepsilon_{k-1}] = 0.
\]

Terms of the following form
\[
E \left( \langle X_0, e_j \rangle \langle X_0, e_i \rangle \langle \rho_{I_h} \ldots \rho_{I_p} (\varepsilon_{k-1}), e_j \rangle \langle \rho_{I_h} \ldots \rho_{I_p} (X_0), e_i \rangle \right)
\]
also vanish due to independence properties, in a similar way as in (15).

For \( p = k \), consider the terms of the form
\[
E \left( \langle X_0, e_j \rangle \langle X_0, e_i \rangle \langle \rho_{I_h} \ldots \rho_{I_p} (\varepsilon_{p-1}), e_j \rangle \langle \rho_{I_h} \ldots \rho_{I_p} (X_0), e_i \rangle \right)
\]
as a result of independence properties. But
\[
\langle C(e_j), e_i \rangle = E \left( \langle X_h, e_j \rangle \langle X_h, e_i \rangle \right)
= E \left[ \langle \varepsilon_h, e_j \rangle \langle \varepsilon_h, e_i \rangle \right] + \ldots + E \left( \langle \rho_{I_h} \ldots \rho_{I_p} (\varepsilon_1), e_j \rangle \langle \rho_{I_h} \ldots \rho_{I_p} (\varepsilon_1), e_i \rangle \right)
+ E \left( \langle \rho_{I_h} \ldots \rho_{I_1} (X_0), e_j \rangle \langle \rho_{I_h} \ldots \rho_{I_1} (X_0), e_i \rangle \right)
\]
by (3), the other terms involved in this relation yield vanishing expectations by use of techniques of the same kind as seen before. Accordingly,
\[
\sum_p E \left( \langle \rho_{I_h} \ldots \rho_{I_p} (\varepsilon_{p-1}), e_j \rangle \langle \rho_{I_h} \ldots \rho_{I_p} (\varepsilon_{p-1}), e_i \rangle \right)
= \langle C(e_j), e_i \rangle - E \left( \langle \rho_{I_h} \ldots \rho_{I_1} (X_0), e_j \rangle \langle \rho_{I_h} \ldots \rho_{I_1} (X_0), e_i \rangle \right).
\]

Thus,
\[
E \left( Z_0, Z_h \right) = \sum_{j,l} E \left( \langle X_0, e_j \rangle \langle X_0, e_l \rangle \langle \rho_{I_h} \ldots \rho_{I_1} (X_0), e_j \rangle \langle \rho_{I_h} \ldots \rho_{I_1} (X_0), e_l \rangle \right)
- \sum_{j,l} \langle C(e_j), e_l \rangle E \left( \langle \rho_{I_h} \ldots \rho_{I_1} (X_0), e_j \rangle \langle \rho_{I_h} \ldots \rho_{I_1} (X_0), e_l \rangle \right).
\]
Let us find upper bounds for each sum in absolute value. For the first one, note that
\[
\sum_{j,l} E \left( |\langle X_0, e_j \rangle \langle X_0, e_l \rangle \rho_{I_h} \cdots \rho_{I_1} (X_0), e_j \rangle \langle \rho_{I_h} \cdots \rho_{I_1} (X_0), e_l \rangle | \right) \\
= \sum_{j,l} E \left( (A_{j, l_1} \cdots A_{j, l_t}) (A_{l_1} \cdots A_{l_t}) (X_0, e_j)^2 \langle X_0, e_l \rangle^2 \sigma (I_1, ..., I_h) \right) \\
= \sum_{j,l} E \left( (A_{j, l_1} \cdots A_{j, l_t}) (A_{l_1} \cdots A_{l_t}) E \left( (X_0, e_j)^2 \langle X_0, e_l \rangle^2 \sigma (I_1, ..., I_h) \right) \right) \\
= \sum_{j,l} E \left( (A_{j, l_1} \cdots A_{j, l_t}) (A_{l_1} \cdots A_{l_t}) E \left( (X_0, e_j)^2 \langle X_0, e_l \rangle^2 \right) \right) \\
= \sum_{j,l} E \left( |(A_{j, l_1} \cdots A_{j, l_t}) (A_{l_1} \cdots A_{l_t})| E \left( (X_0, e_j)^2 \langle X_0, e_l \rangle^2 \right) \right) \\
\leq \left( 1 - q \| \rho_0 \|^2 + q \| \rho_1 \|^2 \right)^h \sum_{j,l} E \left( (X_0, e_j)^2 \langle X_0, e_l \rangle^2 \right) \\
\leq \left( 1 - q \| \rho_0 \|^2 + q \| \rho_1 \|^2 \right)^h E \| X_0 \|^4.
\]

For the second one, we reason as previously and obtain
\[
E \left( |\langle X_0, e_j \rangle \langle X_0, e_l \rangle \rho_{I_h} \cdots \rho_{I_1} (X_0), e_j \rangle \langle \rho_{I_h} \cdots \rho_{I_1} (X_0), e_l \rangle | \right) \\
= E \left( |(A_{j, l_1} \cdots A_{j, l_t}) (A_{l_1} \cdots A_{l_t}) \langle X_0, e_j \rangle \langle X_0, e_l \rangle | \right) \\
= E \left( |(A_{j, l_1} \cdots A_{j, l_t}) (A_{l_1} \cdots A_{l_t}) \langle X_0, e_j \rangle \langle X_0, e_l \rangle | \right) \\
= E \left( |(A_{j, l_1} \cdots A_{j, l_t}) (A_{l_1} \cdots A_{l_t}) | (C(e_j), e_l) \right).
\]

Hence,
\[
\sum_{j,l} \langle C(e_j), e_l \rangle E \left( (\rho_{I_h} \cdots \rho_{I_1} (X_0), e_j \rangle \langle \rho_{I_h} \cdots \rho_{I_1} (X_0), e_l \rangle) \right) \\
= \sum_{j,l} \langle C(e_j), e_l \rangle^2 E \left( |(A_{j, l_1} \cdots A_{j, l_t}) (A_{l_1} \cdots A_{l_t}) | \right) \\
\leq \left( 1 - q \| \rho_0 \|^2 + q \| \rho_1 \|^2 \right)^h \| C \|^2_H.
\]

But
\[
\| C \|^2_H \leq \| C \|^2_{\mathcal{A}} = \left( E \| X_0 \|^2 \right)^2,
\]
so setting \( \alpha_2 = \left( 1 - q \| \rho_0 \|^2 + q \| \rho_1 \|^2 \right)^h \)
\[
| E \langle Z_0, Z_h \rangle_{\mathcal{S}} | \leq \alpha_2^h \left( E \| X_0 \|^4 + \left( E \| X_0 \|^2 \right)^2 \right) = M \alpha_2^h.
\]
Consequently, as $\alpha_2 < 1$,
\[
\left| 2 \sum_{h=1}^{p-1} (p-h) E \langle Z_0, Z_h \rangle \right| \leq 2M \sum_{h=1}^{p-1} (p-h) \alpha_2^h \\
\leq 2Mp \sum_{h=1}^{p-1} = 2Mp \alpha_2 (1 - \alpha_2) \\
\leq 2Mp \frac{\alpha_2}{\alpha_2 - 1}.
\]

In conclusion,
\[
E \| Z_n + \ldots + Z_{n+p-1} \|^2 = O(p),
\]
and therefore we get directly (12), and from [5, Cor. 2.3] we obtain (13). □

The ultimate goal is to estimate the autocorrelation operators $\rho_0$ and $\rho_1$. For that purpose, we can not use the following relation (see [5, Th. 3.2]) which holds in the case of ARH(1) processes:
\[ D = \rho C, \]
where $D$ is the cross-covariance operator, as there are now two choices for $\rho$. This is why we define two cross-covariance operators $D_{i,n}$ enabling us to use relations of the same kind. Let us set, for $i = 0, 1$
\[ D_i(.) = E [ 1_{I_i=1} (X_{0,\cdot}) X_1 ], \ i = 0, 1. \]

This operator corresponds to the cross-covariance operator, but only in the case where the evolution from $X_n$ to $X_{n+1}$ is carried out through the use of $\rho_i$. Note that for all $x \in H$,
\[ D_i(x) = E [ 1_{I_i=1} (X_0, x) (\rho_i (X_0) + \varepsilon_1) ] \\
= E [ 1_{I_i=1} (X_0, x) \rho_i (X_0) ] + E [ 1_{I_i=1} (X_0, x) \varepsilon_1 ] \\
= E [ E [ 1_{I_i=1} (X_0, x) \rho_i (X_0) | I_1 ] ] + E [ E [ 1_{I_i=1} (X_0, x) \varepsilon_1 | I_1 ] ] \\
= E [ 1_{I_i=1} E [ (X_0, x) \rho_i (X_0) | I_1 ] ] + E [ 1_{I_i=1} E [ (X_0, x) \varepsilon_1 | I_1 ] ] \\
= E [ 1_{I_i=1} \rho_i E [ (X_0, x) X_0 | I_1 ] ] + E [ 1_{I_i=1} E [ (X_0, x) \varepsilon _1 ] ]
\]
using Proposition 3.1 for the first term and independence properties for the second. Hence,
\[ D_i(x) = E [ 1_{I_i=1} \rho_i E [ (X_0, x) X_0 ] ] \\
= E [ 1_{I_i=1} \rho_i C(x) ] \\
= P(I_1 = i) \rho_i C(x), \]
and setting $q_i = P(I_1 = i)$, the relation comes out
\[ D_i = q_i \rho_i C \quad (16) \]

The empirical estimator of $D_i$ is
\[ D_{i,n}(x) = \frac{1}{n-1} \sum_{t=1}^{n-1} 1_{I_{t+1}=i} (X_{t, \cdot}) X_{t+1}, \]
In order to obtain the consistency, the assumptions will be stronger than those used for the covariance operator: we will assume that the $X_t$ are bounded for purely technical reasons.

14
Theorem 4.2 Under (H1) and (H2), under all the assumptions of Theorem 2.1, if there is an $A > 0$ such that
\[ \| X_t \| < A, \ t \in \mathbb{Z}, \]
and if
\[ \alpha_2 := (1-q) \| \rho_0 \|^2 + q \| \rho_1 \|^2 < 1, \]
then
\[ E \| D_{n,i} - D_i \|_S^2 = O \left( \frac{1}{n} \right), \ i = 0, 1 \] \hspace{1cm} (17)
and
\[ \| D_{n,i} - D_i \|_S \to 0 \ a.s. \ , i = 0, 1. \] \hspace{1cm} (18)

Proof. Fix $i$. Set
\[ Z'_t = 1_{t+1=i} \langle X_{t+1} , \cdot \rangle X_{t+1} - D_t \]
\[ = 1_{t+1=i} \langle X_{t+1} , \cdot \rangle \varepsilon_{t+1} + 1_{t+1=i} \langle X_{t+1} , \cdot \rangle \rho_t (X_t) - q_t \rho_t C \]
by relation (16) So
\[ \frac{1}{n-1} \sum_{t=1}^{n-1} Z'_t \]
\[ = \frac{1}{n-1} \sum_{t=1}^{n-1} 1_{t+1=i} \langle X_{t+1} , \cdot \rangle \varepsilon_{t+1} + \frac{1}{n-1} \sum_{t=1}^{n-1} 1_{t+1=i} \langle X_{t+1} , \cdot \rangle \rho_t (X_t) - q_t \rho_t C. \]
But the $1_{t+1=i} \langle X_{t+1} , \cdot \rangle \varepsilon_{t+1}$ are orthogonals in $L^2_S$ because, for $t < t'$ without loss of generality,
\[ E \langle 1_{t+1=i} \langle X_{t+1} , \cdot \rangle \varepsilon_{t+1} , 1_{t'+1=i} \langle X_{t'+1} , \cdot \rangle \varepsilon_{t'+1} \rangle_S \]
\[ = E \sum_{i,j} 1_{t+1=i} 1_{t'+1=i} \langle X_{t+1} , e_i \rangle \varepsilon_{t+1} \langle X_{t' +1} , e_j \rangle \varepsilon_{t' +1} \langle X_{t' +1} , e_j \rangle \]
The inversion of the sum and the expectation is possible since
\[ \sum_{i,j} E \left| 1_{t+1=i} 1_{t'+1=i} \langle X_{t+1} , e_i \rangle \varepsilon_{t+1} \langle X_{t' +1} , e_j \rangle \varepsilon_{t' +1} \langle X_{t' +1} , e_j \rangle \right| \]
\[ \leq \frac{1}{2} \left[ \sum_{i,j} E \langle X_{t+1} , e_i \rangle^2 \varepsilon_{t+1}^2 \langle X_{t' +1} , e_j \rangle^2 + \sum_{i,j} E \langle X_{t' +1} , e_i \rangle^2 \varepsilon_{t' +1}^2 \langle X_{t' +1} , e_j \rangle^2 \right] \]
\[ = \frac{1}{2} \left[ 2E \| X_0 \|^2 \| \varepsilon_0 \|^2 \right] \leq \frac{1}{2} E \left[ \| X_0 \|^4 + \| \varepsilon_0 \|^4 \right] < \infty. \]
Consequently,
\[ E \langle 1_{t+1=i} \langle X_{t+1} , \cdot \rangle \varepsilon_{t+1} , 1_{t'+1=i} \langle X_{t'+1} , \cdot \rangle \varepsilon_{t'+1} \rangle_S \]
\[ = \sum_{i,j} E \left[ 1_{t+1=i} 1_{t'+1=i} \langle X_{t+1} , e_i \rangle \varepsilon_{t+1} \langle X_{t' +1} , e_j \rangle \varepsilon_{t' +1} \langle X_{t' +1} , e_j \rangle \right] \]
\[ = \sum_{i,j} E \left[ 1_{t+1=i} 1_{t'+1=i} \langle X_{t+1} , e_i \rangle \varepsilon_{t+1} \langle X_{t' +1} , e_j \rangle \varepsilon_{t' +1} \langle X_{t' +1} , e_j \rangle \right] E \left[ \varepsilon_{t'+1} e_j \right] \]
\[ = 0. \]
The second term of the last right-hand side of (20) is merely equal in $L_\infty$-norm to

$$E \left\| \frac{1}{n-1} \sum_{i=1}^{n-1} 1_{i_{t+1}} \langle X_{t_i} \rangle \varepsilon_{t+1} \right\|^2_s \leq \frac{1}{n-1} E \|X_0\|^2 \|\varepsilon_0\|^2 = O \left( \frac{1}{n} \right).$$

With regard to the second and third terms of (19), we can simply write down the following identity

$$1_{i_{t+1}} \langle X_{t_i} \rangle \rho_t (X_t) = 1_{i_{t+1}} \langle X_{t_i} \rangle \rho_t (X_t) \text{ a.s.,}$$

which yields

$$E \left\| \frac{1}{n-1} \sum_{i=1}^{n-1} 1_{i_{t+1}} \langle X_{t_i} \rangle \rho_t (X_t) - q_i \rho_t C \right\|^2_s = \frac{1}{n-1} E \left\| \frac{1}{n-1} \sum_{i=1}^{n-1} 1_{i_{t+1}} \langle X_{t_i} \rangle X_t - q_i C \right\|^2_s = \frac{1}{n-1} \sum_{i=1}^{n-1} q_i \langle X_{t_i} \rangle X_t - q_i C$$

$$= \rho_t \left[ \frac{1}{n-1} \sum_{i=1}^{n-1} 1_{i_{t+1}} \langle X_{t_i} \rangle \right] \langle X_t \rangle - q_i C$$

$$= \rho_t \left[ \frac{1}{n-1} \sum_{i=1}^{n-1} 1_{i_{t+1}} \langle X_{t_i} \rangle X_t - \frac{1}{n-1} \sum_{i=1}^{n-1} q_i \langle X_{t_i} \rangle X_t \right] \rho_t [X_t]$$

$$+ \rho_t \left[ \frac{1}{n-1} \sum_{i=1}^{n-1} q_i \langle X_{t_i} \rangle X_t - q_i C \right].$$

The second term of the last right-hand side of (20) is merely equal in $L_2^2$-norm to

$$q_i E \|\rho_t (C_n - C)\|_s^2 = O \left( \frac{1}{n} \right)$$

by Theorem 4.1. The first term in $L_2^2$-norm is equal to

$$E \left\| \rho_t \left( \frac{1}{n-1} \sum_{i=1}^{n-1} (1_{i_{t+1}} - q_i) \langle X_{t_i} \rangle X_t \right) \right\|^2_s$$

$$= E \left\| \frac{1}{n-1} \sum_{i=1}^{n-1} (1_{i_{t+1}} - q_i) \langle X_{t_i} \rangle \rho_t (X_t) \right\|^2_s \leq \frac{1}{(n-1)^2} E \left( \sum_{i=1}^{n-1} |1_{i_{t+1}} - q_i| \|\langle X_{t_i} \rangle \rho_t (X_t)\|_s^2 \right)$$

$$\leq \frac{1}{(n-1)^2} E \left( \sum_{i=1}^{n-1} |1_{i_{t+1}} - q_i| \|\rho_t \|_2 \|X_t\|^2 \right) \leq \frac{\|\rho_t\|_2^2 A_4}{(n-1)^2} \sum_{i=1}^{n-1} \left| 1_{i_{t+1}} - q_i \right|^2 \leq \frac{\|\rho_t\|_2^2 A_4}{(n-1)^2} \sum_{i=1}^{n-1} \left( \sum_{i=1}^{n-1} |1_{i_{t+1}} - q_i| \right)^2$$

$$= \frac{\|\rho_t\|_2^2 A_4}{(n-1)^2} \sum_{i=1}^{n-1} E \left[ |1_{i_{t+1}} - q_i|^2 \right] = \frac{\|\rho_t\|_2^2 A_4}{(n-1)^2} E \left[ \sum_{i=1}^{n-1} |1_{i_{t+1}} - q_i|^2 \right].$$

16
because the random variables $|1_{t+1=i} - q_i|$ are i.i.d. and centered, therefore

$$E \left\| \frac{1}{n-1} \sum_{t=1}^{n-1} Z'_{t} \right\|_{s}^{2} = O \left( \frac{1}{n} \right),$$

which implies straightforward (17), and (18) with the help of [5, Cor. 2.3] but arguing in a similar way about $Z'_{n+1} + ... + Z'_{n+p-1}$. 

Finally, with the outcomes of this section, it is possible to slightly modify the results of [12] in order to obtain rates of convergence - such as $n^{-1/4}$ in the general case - of an estimator denoted by $\hat{\rho}_{n,i,a}$ of $\rho_i$ (for $i = 0, 1$). As the technique is similar, this topic is omitted.

References


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<table>
<thead>
<tr>
<th>N°1</th>
<th>Janvier 2002</th>
<th>Functional asymptotic normality of the L2-deviation of the kernel density estimation indexed by class of weight functions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>\textit{Fateh CHEBANA}</td>
</tr>
<tr>
<td>N°2</td>
<td>Janvier 2002</td>
<td>Optimal sampling for density estimation in continuous time</td>
</tr>
<tr>
<td></td>
<td></td>
<td>\textit{Delphine BLANKE et Besnik PUMO}</td>
</tr>
<tr>
<td>N°3</td>
<td>Février 2002</td>
<td>Doubly Stochastic Hilbertian processes</td>
</tr>
<tr>
<td></td>
<td></td>
<td>\textit{Serge GUILLAS}</td>
</tr>
</tbody>
</table>